Chapter 6

The Quantum Delta-Kicked Harmonic Oscillator

6.1 Introduction

Having gone into considerable detail on the dynamics of the classical delta-kicked harmonic oscillator in Chapter 4, and having had a brief introduction to some of the ideas of quantum chaos in Chapter 5, it is now time to study the quantum dynamics of our chosen system. The quantum delta-kicked harmonic oscillator has been studied in some detail by various authors [1, 2, 3], although not to the same extent as the classical case.

In particular work on the localization properties of the delta-kicked harmonic oscillator, expected in the case of an irrational number of kicks per oscillator period [2], is somewhat scarce; more analytical effort appears to have been concentrated on the paradigmatic delta-kicked rotor [4, 5, 6]. As pointed out by Frasca [2], in the case of an irrational number of kicks per oscillation period, the expected localization can be explained qualitatively in much the same way as for the delta-kicked rotor, except that the localization is in energy rather than angular momentum (in both cases these are the action variables in the unperturbed system, so this is perhaps not unexpected). This is by analogy with Anderson localization [7]; the connection between this and localization in the delta-kicked rotor was determined by Grempel, Prange, and Fishman [4]. A comprehensive overview of localization lore is given in [6].

In this Chapter however, as in Chapter 4 describing the classical delta-kicked harmonic oscillator, we will concentrate on the case of a rational number of kicks per oscillation period, where the effects are quite different.

6.2 Floquet Operator

We consider now the dynamics produced by exactly the same Hamiltonian as in Eq. (4.1), but described in terms of operators:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} + K\cos(k\hat{x})\sum_{n=-\infty}^{\infty}\delta(t-n\tau). \tag{6.1}$$

The corresponding Schrödinger wave equation is therefore

$$i\hbar\frac{d}{dt}|\psi\rangle = \left\{\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) + K\cos[\eta(\hat{a}^\dagger+\hat{a})]\sum_{n=-\infty}^{\infty}\delta(t-n\tau)\right\}|\psi\rangle, \tag{6.2}$$

where it is convenient to express the Hamiltonian operator in terms of the harmonic oscillator creation and annihilation operators, $\hat{a}$ and $\hat{a}^\dagger$, defined in terms of the position and momentum operators as

$$\hat{a} = \hat{x}\sqrt{\frac{m\omega}{2\hbar}} + i\hat{p}\sqrt{\frac{1}{2m\hbar\omega}} \tag{6.3}$$

where the creation operator $\hat{a}^\dagger$ is simply the Hermitian conjugate. Note that we have introduced the Lamb-Dicke parameter η, defined as

$$\eta = k\sqrt{\frac{\hbar}{2m\omega}}, \tag{6.4}$$

which turns out to be a significant quantity when considering the transition from quantum to classical chaotic dynamics in this system.

In a similar fashion to the classical case, one can immediately describe the time evolution of the wave-vector between kicks as that of the normal quantum harmonic oscillator:

$$|\psi(t)\rangle = e^{-i(\hat{a}^\dagger\hat{a}+1/2)\omega t}|\psi(0)\rangle. \tag{6.5}$$

For when we do wish to take the kicks into account, we integrate over an infinitesimal interval around the time when a kick occurs:

$$\begin{aligned} i\hbar\int_{n'\tau-\epsilon}^{n'\tau+\epsilon} dt\frac{d}{dt}|\psi(t)\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\int_{n'\tau-\epsilon}^{n'\tau+\epsilon} dt|\psi(t)\rangle \\ + K\cos[\eta(\hat{a}^\dagger+\hat{a})]\int_{n'\tau-\epsilon}^{n'\tau+\epsilon} dt\sum_{n=-\infty}^{\infty}\delta(t-n\tau)|\psi(t)\rangle. \end{aligned} \tag{6.6}$$

As we let $\epsilon \to 0$, the ordinary harmonic oscillator term disappears. If we thus restrict ourselves to integrating over this infinitesimal interval we can ignore this term, and rewrite the Schrödinger equation to and we are left with

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = K\cos[\eta(\hat{a}^\dagger+\hat{a})]\delta(t-n'\tau)|\psi(n'\tau)\rangle. \tag{6.7}$$

We model the delta function with a step function of width ϵ (which is the interval we are integrating over) and height $1/\epsilon$ [8]:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \frac{K}{\epsilon}\cos[\eta(\hat{a}^\dagger + \hat{a})]|\psi(n'\tau)\rangle. \tag{6.8}$$

This differential equation has the obvious solution

$$|\psi(n'\tau + \Delta t)\rangle = e^{-iK\cos[\eta(\hat{a}^\dagger+\hat{a})]\Delta t/\hbar\epsilon}|\psi(n'\tau)\rangle. \tag{6.9}$$

Remembering that we are in fact integrating over a time interval of length ϵ, we let $\Delta t = \epsilon$ and allow $\epsilon \to 0$. We thus have

$$|\psi(n'\tau^+)\rangle = e^{-iK\cos[\eta(\hat{a}^\dagger+\hat{a})]}|\psi(n'\tau^-)\rangle. \tag{6.10}$$

Combining this with the standard harmonic oscillator evolution between kicks, we can write an evolution operator covering a discrete time period, or *Floquet* operator, which maps the wave-vector from just before one kick to just before the next:

$$\hat{F} = e^{-i(\hat{a}^\dagger\hat{a}+1/2)\omega\tau}e^{-iK\cos[\eta(\hat{a}+\hat{a}^\dagger)]/\hbar}. \tag{6.11}$$

This operator applied to a given wave-vector is the exact quantum wave-dynamic equivalent to the mapping described by Eqs. (4.8) and (4.9). Evolution to any point in time of the wave-vector is determined by repeated application of the Floquet operator, possibly followed by application of the harmonic evolution operator $e^{-i(\hat{a}^\dagger\hat{a}+1/2)\omega t}$ for some time $t < \tau$.

6.3 Scaling

If we substitute in the expressions for the dimensionless kick strength $\kappa = k^2K/\sqrt{2}$, and the dimensionless time step $\tau' = \omega\tau$ used in the rescaled classical mapping of Eqs. (4.12,4.13), the Floquet operator can be re-expressed as

$$\hat{F} = e^{-i(\hat{a}^\dagger\hat{a}+1/2)\tau'}e^{-i\kappa\cos[\eta(\hat{a}+\hat{a}^\dagger)]/\sqrt{2}\eta^2}. \tag{6.12}$$

It is this form of the Floquet operator that we will in general use from now on, where the prime will be dropped from τ', for convenience.

We see that this expression of the Floquet operator has three free dimensionless parameters rather than the two observed in the classical mapping; there is additionally η, the Lamb-Dicke parameter. Admittedly we are not making use of the rescaled position and momentum defined in Eqs. (4.10,4.11), but it is clear that no amount of rescaling these observables will remove the $1/\eta^2$ term from the kick operator, while simultaneously preserving the form of κ.

The presence of η in the time evolution operator in fact arises directly from the presence of $\hbar$ in the Schrödinger equation, and can thus be considered a measure of the *classicality* of the dynamics. This can be seen to be reasonable when one considers that a small value of η can be interpreted as meaning that the harmonic oscillator ground state is small when compared to the wavelength of the cosine potential. As $\eta \rightarrow 0$, the harmonic ground state tends to a point compared to a single wavelength of the cosine. Alternatively, one can say that a quantum mechanical treatment of a single particle explicitly precludes the possibility of it having exactly defined values of both position and momentum, as stated by Heisenberg's uncertainty principle [9]. Thus one has to expect to include an extra quantity in the dynamics to take account of the particle's finite size in position and/or momentum space.

As has already been stated, it is convenient to express the all operators, including the Floquet operator, in terms of creation and annihilation operators. In this picture one can define natural harmonic position and momentum operators, as

$$\hat{x}_h = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \tag{6.13}$$

$$\hat{p}_h = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}. \tag{6.14}$$

One can then describe the effect of the kick operator for various η as follows: in the argument of the exponential, it can be seen that κ is divided by η^2, meaning that as η increases, the effect of the kick shrinks. However, the size of the structure in phase space is set by $\cos[\eta(\hat{a} + \hat{a}^\dagger)]$, from which we can see that as η increases, the basic "cell size" of the phase space must also shrink. Classically this would not matter, as it would simply be a case of a point particle being kicked to a lesser degree to get to an exactly equivalent (but closer) region of phase space. This is all scaled out in the dimensionless position and momentum variables used in Chapter 4 and defined by Eqs. (4.10) and (4.11), which is why those particular variables are chosen. From our present point of view however, the size of the initial quantum state remains fixed, as the phase space structure grows or shrinks around it, and again it is obvious that this must be accounted for; hence the extra parameter.

6.4 Coherent States as Initial Conditions

As an equivalent quantum initial condition to classical point particle in phase space, a natural choice is a coherent state, which is a minimum uncertainty state symmetrically localized in both position and momentum space. It could be argued that a more appropriate classical equivalent to such a state would be a Liouville distribution, which in this case would be Gaussian in both position and momentum. As Liouville dynamics can be described by an ensemble of noninteracting classical point particles all undergoing Hamiltonian dynamics [10], this is for our purposes a minor point.

As it stands, if we do wish to explore the dynamics of the delta-kicked harmonic oscillator in a localized region of the classical phase space, then a coherent state is about the best we can do. In this case the initial state equivalent to the classical initial condition (x, p), where x and p are scaled according to Eqs. (4.10) and (4.11), is $|\alpha\rangle$, where $\alpha = \eta(x + ip)/\sqrt{2}$ [11].

If we do choose the initial state to be a coherent state, it is then comparatively simple to determine an analytic formula for the state after successive applications of the Floquet operator, purely in terms of many different coherent states

We first note that

$$e^{-i\kappa\cos[\eta(\hat{a}+\hat{a}^\dagger)]/\sqrt{2}\eta^2} = \sum_{j=-\infty}^{\infty}(-i)^j J_j(\kappa/\sqrt{2}\eta^2)e^{ij\eta(\hat{a}+\hat{a}^\dagger)}, \tag{6.15}$$

where the J_j are Bessel functions [12]. Now, it is clear that the $e^{ij\eta(\hat{a}+\hat{a}^\dagger)}$ are just unitary displacement operators, and using the simple identities $D(\alpha)|\beta\rangle = e^{(\alpha\beta^*-\alpha^*\beta)/2}|\alpha+\beta\rangle$ and $e^{-i\hat{a}^\dagger\hat{a}\tau}|\gamma\rangle = |\gamma e^{-i\omega\tau}\rangle$ where $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$ are coherent states [11], we can easily determine that

$$\hat{F}|\alpha\rangle = \sum_{j=-\infty}^{\infty}(-i)^j J_j(\kappa/\sqrt{2}\eta^2)e^{ij\eta(\alpha+\alpha^*)/2}|(\alpha+ij\eta)e^{-i2\pi r/q}\rangle. \tag{6.16}$$

Applying this recursively brings us to

$$\begin{aligned}\hat{F}^N|\alpha\rangle = \sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_N=-\infty}^{\infty}(-i)^{\sum_{k=1}^N j_k}\prod_{k=1}^{N}J_{j_k}(\kappa/\sqrt{2}\eta^2)\\ \times e^{i\eta\{j_1|\alpha|\cos\theta+\sum_{l=1}^N j_l|\alpha|\cos[\theta+(k-1)2\pi r/q]-\sum_{l=1}^N\sum_{m=1}^{l-1}j_m\cos[(l-m)2\pi r/q]\}}\\ \times\left|\alpha e^{-i2\pi Nr/q}+i\eta\sum_{n=1}^{N}j_n e^{-i(N+1-j)2\pi r/q}\right\rangle,\end{aligned} \tag{6.17}$$

where θ is the phase angle of α, and the expressions inside the kets are complex numbers describing where the coherent states the kets represent are centred in phase space [11].

Although Eq. (6.17) is an analytic solution to the time evolution of a coherent state, the effort needed to actually determine a specific example is obviously going to grow rapidly with the number of kicks applied. It is generally easier just to determine the time evolution numerically.

6.5 Quantum Translational Symmetry

In the case of the classical delta-kicked harmonic oscillator, there was a crystal symmetry in the phase space where $q \in q_c = \{1, 2, 3, 4, 6\}$. We naturally expect some kind of translational symmetry to occur in the quantum case also. The analytic investigation presented here is based on the treatment of Borgonovi and Rebuzzini [1] . We begin by considering the unitary displacement operator [11],

$$D(\alpha) = e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}} = e^{i(\varpi\hat{x}-\xi\hat{p})}, \tag{6.18}$$

which is the operator which creates the coherent state $|\alpha\rangle$ from the vacuum. The position and momentum operators $\hat{x}$, $\hat{p}$, are scaled in dimensionless harmonic units, as defined in Eqs. (6.13) and (6.13) (we have dropped the subscript h for convenience), and

$\alpha = (\xi + i\varpi)/\sqrt{2}$. The displacement operator acting on a wavefunction is a natural quantum analogue to the act of translating a classical point particle in phase space, and it is by transformations of this kind that we determined translational symmetry in the classical phase space in Chapter 4. We now consider the Floquet operator $\hat{F}$ of Eq. (6.12), and determine the commutation properties of it with the displacement operator.

Taking first the part of the Floquet operator which determines the harmonic evolution between kicks, we again make use of $D(\alpha)|\beta\rangle = e^{(\alpha\beta^* - \alpha^*\beta)/2}|\alpha + \beta\rangle$ and $e^{-i\hat{a}^\dagger\hat{a}\tau}|\gamma\rangle = |\gamma e^{-i\omega\tau}\rangle$ [11]. From this it is easy to see that

$$\begin{aligned} D(\alpha)e^{-i\hat{a}^\dagger\hat{a}\tau}|\beta\rangle &= e^{(\alpha\beta^* e^{i\tau} - \alpha^*\beta e^{-i\tau})/2}|\alpha + \beta e^{-i\tau}\rangle \\ &= e^{(\alpha e^{i\tau}\beta^* - \alpha^* e^{-i\tau}\beta)/2}|(\alpha e^{i\tau} + \beta)e^{-i\tau}\rangle \\ &= e^{-i\hat{a}^\dagger\hat{a}\tau}D(\alpha e^{i\tau})|\beta\rangle, \end{aligned} \tag{6.19}$$

which implies directly that

$$D(\alpha)e^{-i\hat{a}^\dagger\hat{a}\omega\tau} = e^{-i\hat{a}^\dagger\hat{a}\omega\tau}D(\alpha e^{i\omega\tau}). \tag{6.20}$$

This leaves us with the kicking part of the Floquet operator, which we express in terms of the harmonically scaled position operator as $e^{-i\cos(\eta\sqrt{2}\hat{x})/\sqrt{2}\eta^2}$. We make use of the fact that $e^{-i\xi\hat{p}}|a\rangle = |a + \xi\rangle$ [9], where $|a\rangle$ is an eigenstate of the position operator $\hat{x}$, with eigenvalue a. From this we get

$$\begin{aligned} e^{-i\xi\hat{p}}e^{-i\cos(\eta\sqrt{2}\hat{x})/\sqrt{2}\eta^2}|a\rangle &= e^{-i\cos(\eta\sqrt{2}a)/\sqrt{2}\eta^2}|a + \xi\rangle \\ &= e^{-i\cos[\eta\sqrt{2}(\hat{x}-\xi)]/\sqrt{2}\eta^2}|a + \xi\rangle \\ &= e^{-i\cos[\eta\sqrt{2}(\hat{x}-\xi)]/\sqrt{2}\eta^2}e^{-i\xi\hat{p}}|a\rangle, \end{aligned} \tag{6.21}$$

and so, in general

$$e^{-i\xi\hat{p}}e^{-i\cos(\eta\sqrt{2}\hat{x})/\sqrt{2}\eta^2} = e^{-i\cos[\eta\sqrt{2}(\hat{x}-\xi)]/\sqrt{2}\eta^2}e^{-i\xi\hat{p}}. \tag{6.22}$$

From this we get directly, using the Campbell-Baker-Hausdorff identity [9],

$$\begin{aligned} e^{i(\varpi\hat{x}-\xi\hat{p})}e^{-i\cos(\eta\sqrt{2}\hat{x})/\sqrt{2}\eta^2} &= e^{-i\xi\hat{p}}e^{i\varpi\hat{x}}e^{-i\xi\varpi/2}e^{-i\cos(\eta\sqrt{2}\hat{x})/\sqrt{2}\eta^2} \\ &= e^{-i\cos[\eta\sqrt{2}(\hat{x}-\xi)]/\sqrt{2}\eta^2}e^{-i\xi\hat{p}}e^{i\varpi\hat{x}}e^{-i\xi\varpi/2} \\ &= e^{-i\cos[\eta\sqrt{2}(\hat{x}-\xi)]/\sqrt{2}\eta^2}e^{i(\varpi\hat{x}-\xi\hat{p})}. \end{aligned} \tag{6.23}$$

Finally, combining Eqs. (6.20) and (6.23), and expressing everything in terms of α and the creation and annihilation operators, we get

$$D(\alpha)e^{-i(\hat{a}^\dagger\hat{a}+1/2)\tau}e^{-i\kappa\cos[\eta(\hat{a}+\hat{a}^\dagger])/\sqrt{2}\eta^2} = e^{-i(\hat{a}^\dagger\hat{a}+1/2)\tau}e^{-i\kappa\cos[\eta(\hat{a}+\hat{a}^\dagger-\alpha-\alpha^*)]/\sqrt{2}\eta^2}D(\alpha e^{i\tau}), \tag{6.24}$$

and hence, setting $\tau = 2\pi r/q$,

$$D(\alpha)\hat{F}^q = \prod_{j=0}^{q-1}\left\{e^{-i(\hat{a}^\dagger\hat{a}+1/2)2\pi r/q}e^{-i\kappa\cos[\eta(\hat{a}+\hat{a}-\alpha_j-\alpha_j^*)]/\sqrt{2}\eta^2}\right\}D(\alpha), \tag{6.25}$$

where $\alpha_j = \alpha e^{i2\pi jr/q}$. The product of Floquet operators $\hat{F}^q$ corresponds to the mapping of Eq. (4.16) which we used to investigate the symmetry properties of the classical system.

It can be seen that $D(\alpha)$ commutes with $\hat{F}^q$ if

$$\eta(\alpha_j + \alpha_j^*) = \sqrt{2}\eta\xi_j = 2\pi l_j, \quad l_j \in \mathbb{Z} \; \forall \; j. \tag{6.26}$$

From Eq. (6.26) we arrive at, in a similar manner to the derivation of Eq. (4.63) in the classical case,

$$l_j = l_0 \cos(2\pi jr/q) - i\frac{(\alpha - \alpha^*)\eta}{\sqrt{2}\pi} \sin(2\pi jr/q). \tag{6.27}$$

In an exactly analogous manner to the classical case, we arrive at the conclusion that for $D(\alpha)$ to commute with $\hat{F}^q$, the value of q must be such that $q \in q_c$. For the interesting cases of $q = 3$, 4, or 6 one can easily deduce the set of displacement operators that commute with $\hat{F}^q$ [1]. In the case of $q = 4$, we have

$$\exp\left[-i\frac{2\pi}{\eta}\left(\frac{k\hat{x} + l\hat{p}}{\sqrt{2}} + \frac{m\pi}{2\eta}\right)\right]; \; k, l, m \in \mathbb{Z}. \tag{6.28}$$

The phase factor $e^{-i\pi^2 m/\eta^2}$ has been added in order to induce on the set of displacement operators a (non-Abelian) closed group structure. For $q = 3$ or 6, we have

$$\exp\left[-i\frac{2\pi}{\eta}\left(\frac{(k-l)\hat{x}/\sqrt{3} + (k+l)\hat{p}}{\sqrt{2}} + \frac{m\pi}{\eta\sqrt{3}}\right)\right]; \; k, l, m \in \mathbb{Z}, \tag{6.29}$$

which also has an additional phase factor, $e^{-i2\pi^2 m/\eta^2\sqrt{3}}$. Note however that for certain resonance values of η, these additional scalar phase factors have no effect; in such cases the group structure is Abelian [1].

This implies that for $q \in q_c$, the eigenstates of $\hat{F}^q$ are invariant under certain displacements (of which there are nevertheless an infinite number), and are thus extended. Localization is thus not expected to take place, in a similar fashion to the case of quantum resonances in a delta-kicked rotor [2, 4, 13], where the eigenstates are delocalized Bloch waves, and one expects to observe ballistic expansion in momentum space.

References

[1] F. Borgonovi and L. Rebuzzini, Phys. Rev. E **52**, 2302 (1995).

[2] M. Frasca, Phys. Lett. A **231**, 344 (1997).

[3] G. P. Berman, V. Yu. Rubaev, and G. M. Zaslavsky, Nonlinearity **4**, 543 (1991); T. Hogg and B. A. Huberman, Phys. Rev. A **28**, 28 (1983).

[4] D. R. Grempel, S. Fishman, and R. E. Prange, Phys. Rev. Lett. **49**, 833 (1982); D. R. Grempel, R. E. Prange, and S. Fishman, Phys. Rev. A **29**, 1639 (1984).

[5] L. E. Reichl, *The Transition to Chaos In Conservative Classical Systems: Quantum Manifestations* (Springer-Verlag, New York 1992).

[6] F. Haake, *Quantum Signatures of Chaos* (Springer-Verlag, Berlin 1991).

[7] P. W. Anderson, Phys. Rev. **109**, 1492 (1958); P. W. Anderson, Rev. Mod. Phys. **50**, 191 (1978).

[8] A very readable and comprehensive overview of functions with delta-function limits and the properties of delta-functions generally is given in C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (Wiley-Interscience, New York 1977).

[9] A. Galindo and P. Pascual, *Quantum Mechanics* (Springer-Verlag, Berlin 1990).

[10] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading 1980).

[11] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin 1994).

[12] I. S. Gradsteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, (Academic Press, New York 1980).

[13] F. M. Izrailev and D. L. Shepelyanskii, Sov. Phys. Dokl. **24**, 996 (1980); F. M. Izrailev and D. L. Shepelyanskii, Theor. Math. Phys. **43**, 553 (1980).